Inverse Matrices

Definition

An $n \times n$ matrix A is said to be **nonsingular** (or *invertible*) if an $n \times n$ matrix A^{-1} exists with $AA^{-1} = A^{-1}A = I$. The matrix A^{-1} is called the **inverse** of A. A matrix without an inverse is called **singular** (or *noninvertible*).

Theorem

For any nonsingular $n \times n$ matrix A:

- (i) A^{-1} is unique.
- (ii) A^{-1} is nonsingular and $(A^{-1})^{-1} = A$.
- (iii) If *B* is also a nonsingular $n \times n$ matrix, then $(AB)^{-1} = B^{-1}A^{-1}$.

Solving a Linear System Using Inverse Matrix:

 $A\mathbf{x} = \mathbf{b} \implies \mathbf{x} = A^{-1}\mathbf{b}$

Computation of Inverse Matrix:

Let B_j be the *j*th column of the $n \times n$ matrix B,

$$B_j = \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}$$

If AB = C, then the *j*th column of C is given by the product

$$\begin{bmatrix} c_{1j} \\ c_{2j} \\ \vdots \\ c_{nj} \end{bmatrix} = C_j = AB_j = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^n a_{1k} b_{kj} \\ \sum_{k=1}^n a_{2k} b_{kj} \\ \vdots \\ \sum_{k=1}^n a_{nk} b_{kj} \end{bmatrix}$$

Suppose that A^{-1} exists and that $A^{-1} = B = (b_{ij})$. Then AB = I and

$$AB_{j} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$
, where the value 1 appears in the *j*th row.

- Solving the above linear system *j*th column of the inverse matrix is obtained. So,
- To find *B* we need to solve *n* linear systems in which the *j*th column of the inverse is the solution of the linear system with right-hand side the *j*th column of *I*.

Illustration

To determine the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix},$$

let us first consider the product AB, where B is an arbitrary 3×3 matrix.

$$AB = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$
$$= \begin{bmatrix} b_{11} + 2b_{21} - b_{31} & b_{12} + 2b_{22} - b_{32} & b_{13} + 2b_{23} - b_{33} \\ 2b_{11} + b_{21} & 2b_{12} + b_{22} & 2b_{13} + b_{23} \\ -b_{11} + b_{21} + 2b_{31} & -b_{12} + b_{22} + 2b_{32} & -b_{13} + b_{23} + 2b_{33} \end{bmatrix}.$$

If $B = A^{-1}$, then AB = I, so

$$b_{11} + 2b_{21} - b_{31} = 1, \qquad b_{12} + 2b_{22} - b_{32} = 0, \qquad b_{13} + 2b_{23} - b_{33} = 0,$$

$$2b_{11} + b_{21} = 0, \qquad 2b_{12} + b_{22} = 1, \qquad \text{and} \qquad 2b_{13} + b_{23} = 0,$$

$$-b_{11} + b_{21} + 2b_{31} = 0, \qquad -b_{12} + b_{22} + 2b_{32} = 0, \qquad -b_{13} + b_{23} + 2b_{33} = 1.$$

Since for all systems the coefficients matrix is the same, Gaussian elimination can be performed on a larger augmented matrix:

First, performing $(E_2 - 2E_1) \rightarrow (E_2)$ and $(E_3 + E_1) \rightarrow (E_3)$, followed by $(E_3 + E_2) \rightarrow (E_3)$ produces

$$\begin{bmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -3 & 2 & -2 & 1 & 0 \\ 0 & 3 & 1 & 1 & 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -3 & 2 & -2 & 1 & 0 \\ 0 & 0 & 3 & -1 & 1 & 1 \end{bmatrix}.$$

Backward substitution is performed on each of the three augmented matrices,

$$\begin{bmatrix} 1 & 2 & -1 & \vdots & 1 \\ 0 & -3 & 2 & \vdots & -2 \\ 0 & 0 & 3 & \vdots & -1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & -1 & \vdots & 0 \\ 0 & -3 & 2 & \vdots & 1 \\ 0 & 0 & 3 & \vdots & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & -1 & \vdots & 0 \\ 0 & -3 & 2 & \vdots & 0 \\ 0 & 0 & 3 & \vdots & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & -1 & \vdots & 0 \\ 0 & -3 & 2 & \vdots & 0 \\ 0 & 0 & 3 & \vdots & 1 \end{bmatrix},$$

to eventually give

$$b_{11} = -\frac{2}{9}, \qquad b_{12} = \frac{5}{9}, \qquad b_{13} = -\frac{1}{9},$$

$$b_{21} = \frac{4}{9}, \qquad b_{22} = -\frac{1}{9}, \qquad \text{and} \qquad b_{23} = \frac{2}{9},$$

$$b_{31} = -\frac{1}{3}, \qquad b_{32} = \frac{1}{3}, \qquad b_{32} = \frac{1}{3}.$$

So, the inverse matrix is:

$$B = A^{-1} = \begin{bmatrix} -\frac{2}{9} & \frac{5}{9} & -\frac{1}{9} \\ \frac{4}{9} & -\frac{1}{9} & \frac{2}{9} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

Gaussian elimination with backward substitution requires:

$$\frac{4}{3}n^3 - \frac{1}{3}n$$
 multiplications/divisions
$$\frac{4}{3}n^3 - \frac{3}{2}n^2 + \frac{n}{6}$$
 additions/subtractions

to solve the *n* linear systems.

So, it is not computationally efficient to determine A^{-1} in order to solve the system.

Transpose of a Matrix

The **transpose** of an $n \times m$ matrix $A = [a_{ij}]$ is the $m \times n$ matrix $A^t = [a_{ji}]$, where for each i, the *i*th column of A^t is the same as the *i*th row of A. A square matrix A is called **symmetric** if $A = A^t$.

The Determinant of a Matrix

Definition

Suppose that A is a square matrix.

- (i) If A = [a] is a 1×1 matrix, then det A = a.
- (ii) If *A* is an $n \times n$ matrix, with n > 1 the minor M_{ij} is the determinant of the $(n-1) \times (n-1)$ submatrix of *A* obtained by deleting the *i*th row and *j*th column of the matrix *A*.
- (iii) The cofactor A_{ij} associated with M_{ij} is defined by $A_{ij} = (-1)^{i+j} M_{ij}$.
- (iv) The **determinant** of the $n \times n$ matrix A, when n > 1, is given either by

$$\det A = \sum_{j=1}^{n} a_{ij} A_{ij} = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij}, \text{ for any } i = 1, 2, \cdots, n,$$

or by

det
$$A = \sum_{i=1}^{n} a_{ij} A_{ij} = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij}$$
, for any $j = 1, 2, \cdots, n$.

Theorem

Suppose *A* is an $n \times n$ matrix:

- (i) If any row or column of A has only zero entries, then det A = 0.
- (ii) If A has two rows or two columns the same, then det A = 0.
- (iii) If \tilde{A} is obtained from A by the operation $(E_i) \leftrightarrow (E_j)$, with $i \neq j$, then det $\tilde{A} = -\det A$.
- (iv) If \tilde{A} is obtained from A by the operation $(\lambda E_i) \to (E_i)$, then det $\tilde{A} = \lambda \det A$.

- (v) If A is obtained from A by the operation $(E_i + \lambda E_j) \rightarrow (E_i)$ with $i \neq j$, then $\det \tilde{A} = \det A$.
- (vi) If *B* is also an $n \times n$ matrix, then det $AB = \det A \det B$.
- (vii) $\det A^t = \det A$.
- (viii) When A^{-1} exists, det $A^{-1} = (\det A)^{-1}$.
 - (ix) If A is an upper triangular, lower triangular, or diagonal matrix, then $\det A = \prod_{i=1}^{n} a_{ii}$.

Theorem

The following statements are equivalent for any $n \times n$ matrix A:

- (i) The equation $A\mathbf{x} = \mathbf{0}$ has the unique solution $\mathbf{x} = \mathbf{0}$.
- (ii) The system $A\mathbf{x} = \mathbf{b}$ has a unique solution for any *n*-dimensional column vector \mathbf{b} .

(iii) The matrix A is nonsingular; that is, A^{-1} exists.

- (iv) det $A \neq 0$.
- (v) Gaussian elimination with row interchanges can be performed on the system $A\mathbf{x} = \mathbf{b}$ for any *n*-dimensional column vector \mathbf{b} .

HOMEWORK 7:

Exercise Set 6.3: 7 Exercise Set 6.4: 7, 8

Matrix Factorization

$A\mathbf{x} = \mathbf{b}$ A = LU

L is lower triangular U is upper triangular

Not all matrices have this type of representation, but many do that occur in the application of numerical techniques.

 $A\mathbf{x} = \mathbf{b}$ \blacksquare $LU\mathbf{x} = \mathbf{b}$

To compute x, two following systems are solved respectively:

Ly = b

 $U\mathbf{x} = \mathbf{y}$

Determination of L and U Matrices

Gaussian elimination method consists of several steps to make the matrix upper triangular. $\bar{a}_{ii}^{(i)}$ for each i=1,2,...,n is defined as the pivot element

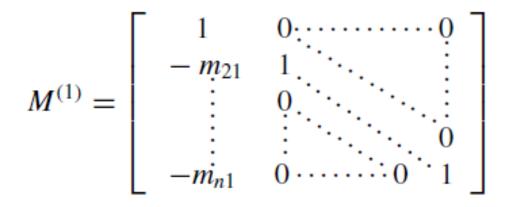
corresponding to the *i*th step. Suppose all pivot elements are non-zero.

The first step in the Gaussian elimination process

$$(E_j - m_{j,1}E_1) \to (E_j), \text{ where } m_{j,1} = \frac{a_{j1}^{(1)}}{a_{11}^{(1)}}.$$
 (6.8)
 $j = 2, 3, \dots, n$

Operations (6.8) can be accomplished by multiplying A on the left by the

matrix:



This is called the **first Gaussian transformation matrix**. We denote the product of this matrix with $A^{(1)} \equiv A$ by $A^{(2)}$ and with **b** by $\mathbf{b}^{(2)}$, so

$$A^{(2)}\mathbf{x} = M^{(1)}A\mathbf{x} = M^{(1)}\mathbf{b} = \mathbf{b}^{(2)}.$$

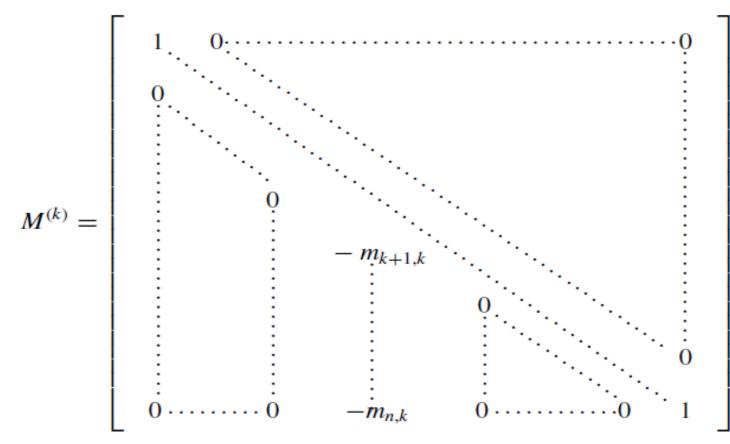
In a similar manner we construct $M^{(2)}$, the identity matrix with the entries below the diagonal in the second column replaced by the negatives of the multipliers

$$m_{j,2} = \frac{a_{j2}^{(2)}}{a_{22}^{(2)}}$$

The product of this matrix with $A^{(2)}$ has zeros below the diagonal in the first two columns, and we let

$$A^{(3)}\mathbf{x} = M^{(2)}A^{(2)}\mathbf{x} = M^{(2)}M^{(1)}A\mathbf{x} = M^{(2)}M^{(1)}\mathbf{b} = \mathbf{b}^{(3)}.$$

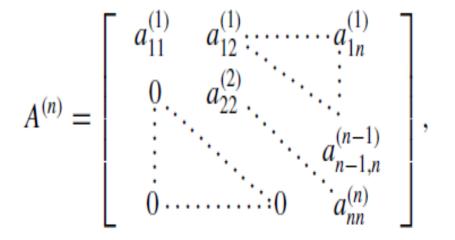
In general, with $A^{(k)}\mathbf{x} = \mathbf{b}^{(k)}$ already formed, multiply by the *k*th Gaussian transformation matrix



to obtain

$$A^{(k+1)}\mathbf{x} = M^{(k)}A^{(k)}\mathbf{x} = M^{(k)}\cdots M^{(1)}A\mathbf{x} = M^{(k)}\mathbf{b}^{(k)} = \mathbf{b}^{(k+1)} = M^{(k)}\cdots M^{(1)}\mathbf{b}.$$
 (6.9)

The process ends with the formation of $A^{(n)}\mathbf{x} = \mathbf{b}^{(n)}$, where $A^{(n)}$ is the upper triangular matrix



given by

$$A^{(n)} = M^{(n-1)} M^{(n-2)} \cdots M^{(1)} A.$$