Inverse Matrices

Definition

An $n \times n$ matrix A is said to be **nonsingular** (or *invertible*) if an $n \times n$ matrix A^{-1} exists with $AA^{-1} = A^{-1}A = I$. The matrix A^{-1} is called the **inverse** of A. A matrix without an inverse is called **singular** (or *noninvertible*). ■

Theorem

For any nonsingular $n \times n$ matrix A:

(i) A^{-1} is unique.

(ii) A^{-1} is nonsingular and $(A^{-1})^{-1} = A$.

(iii) If B is also a nonsingular $n \times n$ matrix, then $(AB)^{-1} = B^{-1}A^{-1}$.

Solving a Linear System Using Inverse Matrix:

$$\mathbf{A}\mathbf{x} = \mathbf{b} \quad \Rightarrow \quad \mathbf{x} = A^{-1}\mathbf{b}$$

Computation of Inverse Matrix:

Let B_j be the jth column of the $n \times n$ matrix B,

$$B_j = \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}$$

If $AB = C$, then the jth column of C is given by the product

$$\begin{bmatrix} c_{1j} \\ c_{2j} \\ \vdots \\ c_{nj} \end{bmatrix} = C_j = AB_j = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^{n} a_{1k} b_{kj} \\ \sum_{k=1}^{n} a_{2k} b_{kj} \\ \vdots \\ \sum_{k=1}^{n} a_{nk} b_{kj} \end{bmatrix}$$

Suppose that A^{-1} exists and that $A^{-1} = B = (b_{ij})$. Then $AB = I$ and

$$AB_j = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$ where the value 1 appears in the jth row.

Solving the above linear system jth column of the inverse matrix is obtained. So,

To find B we need to solve n linear systems in which the jth column of the inverse is the solution of the linear system with right-hand side the jth column of I.

Illustration

To determine the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix},$$

let us first consider the product AB, where B is an arbitrary 3×3 matrix.

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \\ &= \begin{bmatrix} b_{11} + 2b_{21} - b_{31} & b_{12} + 2b_{22} - b_{32} & b_{13} + 2b_{23} - b_{33} \\ 2b_{11} + b_{21} & 2b_{12} + b_{22} & 2b_{13} + b_{23} \\ -b_{11} + b_{21} + 2b_{31} & -b_{12} + b_{22} + 2b_{32} & -b_{13} + b_{23} + 2b_{33} \end{bmatrix}. \end{aligned}$$

If $B = A^{-1}$, then $AB = I$, so

$$\begin{aligned} b_{11}+2b_{21}-\ \ b_{31}&=1, & b_{12}+2b_{22}-\ \ b_{32}&=0, & & b_{13}+2b_{23}-\ \ b_{33}=0,\\ 2b_{11}+\ \ b_{21}\qquad\quad&=0, & 2b_{12}+\ \ b_{22}\qquad\quad&=1, &\text{and}\quad & 2b_{13}+\ \ b_{23}\qquad\quad=0,\\ -b_{11}+\ \ b_{21}+2b_{31}&=0, & -b_{12}+\ \ b_{22}+2b_{32}&=0, & & -b_{13}+\ \ b_{23}+2b_{33}=1. \end{aligned}$$

Since for all systems the coefficients matrix is the same, Gaussian elimination can be performed on a larger augmented matrix:

$$\left[\begin{array}{rrrrr:r} 1 & 2 & -1 & 1 & 0 & 0\\ 2 & 1 & 0 & 0 & 1 & 0\\ -1 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

First, performing $(E_2-2E_1)\to(E_2)$ and $(E_3+E_1)\to(E_3)$, followed by $(E_3+E_2)\to(E_3)$ produces

$$\left[\begin{array}{rrrrr:r} 1 & 2 & -1 & 1 & 0 & 0\\ 0 & -3 & 2 & -2 & 1 & 0\\ 0 & 3 & 1 & 1 & 0 & 1 \end{array}\right] \quad\text{and}\quad \left[\begin{array}{rrrrr:r} 1 & 2 & -1 & 1 & 0 & 0\\ 0 & -3 & 2 & -2 & 1 & 0\\ 0 & 0 & 3 & -1 & 1 & 1 \end{array}\right].$$

Backward substitution is performed on each of the three augmented matrices,

$$\left[\begin{array}{ccc:c} 1 & 2 & -1 & 1 \\ 0 & -3 & 2 & -2 \\ 0 & 0 & 3 & -1 \end{array}\right], \left[\begin{array}{ccc:c} 1 & 2 & -1 & 0 \\ 0 & -3 & 2 & 1 \\ 0 & 0 & 3 & 1 \end{array}\right], \left[\begin{array}{ccc:c} 1 & 2 & -1 & 0 \\ 0 & -3 & 2 & 0 \\ 0 & 0 & 3 & 1 \end{array}\right],$$

to eventually give

$$\begin{array}{llll} b_{11} = -\frac{2}{9}, & b_{12} = \frac{5}{9}, & & b_{13} = -\frac{1}{9}, \\ b_{21} = \frac{4}{9}, & b_{22} = -\frac{1}{9}, & \text{and} & b_{23} = \frac{2}{9}, \\ b_{31} = -\frac{1}{3}, & b_{32} = \frac{1}{3}, & & b_{32} = \frac{1}{3}. \end{array}$$

So, the inverse matrix is:

$$B = A^{-1} = \begin{bmatrix} -\frac{2}{9} & \frac{5}{9} & -\frac{1}{9} \\ \frac{4}{9} & -\frac{1}{9} & \frac{2}{9} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

Gaussian elimination with backward substitution requires:

$$\frac{4}{3}n^3 - \frac{1}{3}n \text{ multiplications/divisions}$$

$$\frac{4}{3}n^3 - \frac{3}{2}n^2 + \frac{n}{6} \text{ additions/subtractions}$$

to solve the n linear systems.

So, it is not computationally efficient to determine A^{-1} in order to solve the system.

Transpose of a Matrix

The **transpose** of an $n \times m$ matrix $A = [a_{ij}]$ is the $m \times n$ matrix $A^t = [a_{ji}]$, where for each i, the ith column of A^t is the same as the ith row of A. A square matrix A is called **symmetric** if $A = A^t$.

The Determinant of a Matrix

Definition

Suppose that A is a square matrix.

(i) If $A = [a]$ is a 1×1 matrix, then $\det A = a$.

(ii) If A is an $n \times n$ matrix, with $n > 1$ the **minor** M_{ij} is the determinant of the $(n-1) \times (n-1)$ submatrix of A obtained by deleting the ith row and jth column of the matrix A.

(iii) The **cofactor** A_{ij} associated with M_{ij} is defined by $A_{ij} = (-1)^{i+j} M_{ij}$.

(iv) The **determinant** of the $n \times n$ matrix A, when $n > 1$, is given either by

$$\det A = \sum_{j=1}^{n} a_{ij} A_{ij} = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij}, \quad \text{for any } i = 1, 2, \cdots, n,$$

or by

$$\det A = \sum_{i=1}^{n} a_{ij}A_{ij} = \sum_{i=1}^{n} (-1)^{i+j} a_{ij}M_{ij}, \quad \text{for any } j = 1, 2, \cdots, n. \qquad \blacksquare$$

Theorem

Suppose A is an $n \times n$ matrix:

(i) If any row or column of A has only zero entries, then $\det A = 0$.

(ii) If A has two rows or two columns the same, then $\det A = 0$.

(iii) If $\tilde{A}$ is obtained from A by the operation $(E_i) \leftrightarrow (E_j)$, with $i \neq j$, then $\det \tilde{A} = -\det\ A$.

(iv) If $\tilde{A}$ is obtained from A by the operation $(\lambda E_i) \rightarrow (E_i)$, then $\det \tilde{A} = \lambda \det A$.

(v) If $\tilde{A}$ is obtained from A by the operation $(E_i + \lambda E_j) \rightarrow (E_i)$ with $i \neq j$, then $\det \tilde{A} = \det A$.

(vi) If B is also an $n \times n$ matrix, then $\det AB = \det A \det B$.

(vii) $\det A^t = \det A$.

(viii) When A^{-1} exists, $\det A^{-1} = (\det A)^{-1}$.

(ix) If A is an upper triangular, lower triangular, or diagonal matrix, then $\det A = \prod_{i=1}^{n} a_{ii}$. ∎

Theorem

The following statements are equivalent for any $n \times n$ matrix A:

(i) The equation $A\mathbf{x} = \mathbf{0}$ has the unique solution $\mathbf{x} = \mathbf{0}$.

(ii) The system $A\mathbf{x} = \mathbf{b}$ has a unique solution for any n-dimensional column vector $\mathbf{b}$.

(iii) The matrix A is nonsingular; that is, A^{-1} exists.

(iv) $\det A \neq 0$.

(v) Gaussian elimination with row interchanges can be performed on the system $A\mathbf{x} = \mathbf{b}$ for any n-dimensional column vector $\mathbf{b}$. ∎

HOMEWORK 7:

Exercise Set 6.3: 7

Exercise Set 6.4: 7, 8

Matrix Factorization

$A\mathbf{x} = \mathbf{b}$ $\qquad$ $A = LU$ $\qquad$ L is lower triangular

U is upper triangular

Not all matrices have this type of representation, but many do that occur in the application of numerical techniques.

$$A\mathbf{x} = \mathbf{b} \quad \Rightarrow \quad \boldsymbol{LU}\mathbf{x} = \mathbf{b}$$

To compute x, two following systems are solved respectively:

$$\boldsymbol{L}\mathbf{y} = \mathbf{b}$$

$$\boldsymbol{U}\mathbf{x} = \mathbf{y}$$

Determination of L and U Matrices

Gaussian elimination method consists of several steps to make the matrix upper triangular. $\bar{a}_{ii}^{(i)}$ for each i=1,2,…,n is defined as the pivot element corresponding to the ith step. Suppose all pivot elements are non-zero.

The first step in the Gaussian elimination process

$$(E_j - m_{j,1}E_1) \to (E_j), \quad \text{where} \quad m_{j,1} = \frac{a_{j1}^{(1)}}{a_{11}^{(1)}}. \tag{6.8}$$

$j = 2, 3, \ldots, n$

Operations (6.8) can be accomplished by multiplying A on the left by the matrix:

$$M^{(1)} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ -m_{21} & 1 & \ddots & & \vdots \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -m_{n1} & 0 & \cdots & 0 & 1 \end{bmatrix}$$

This is called the **first Gaussian transformation matrix**. We denote the product of this matrix with $A^{(1)} \equiv A$ by $A^{(2)}$ and with $\mathbf{b}$ by $\mathbf{b}^{(2)}$, so

$$A^{(2)}\mathbf{x} = M^{(1)}A\mathbf{x} = M^{(1)}\mathbf{b} = \mathbf{b}^{(2)}.$$

In a similar manner we construct $M^{(2)}$, the identity matrix with the entries below the diagonal in the second column replaced by the negatives of the multipliers

$$m_{j,2} = \frac{a_{j2}^{(2)}}{a_{22}^{(2)}}.$$

The product of this matrix with $A^{(2)}$ has zeros below the diagonal in the first two columns, and we let

$$A^{(3)}\mathbf{x} = M^{(2)}A^{(2)}\mathbf{x} = M^{(2)}M^{(1)}A\mathbf{x} = M^{(2)}M^{(1)}\mathbf{b} = \mathbf{b}^{(3)}.$$

In general, with $A^{(k)}\mathbf{x} = \mathbf{b}^{(k)}$ already formed, multiply by the ***k*th Gaussian transformation matrix**

$$M^{(k)} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & & & \vdots \\ \vdots & & 0 & \ddots & \ddots & & & \vdots \\ \vdots & & \vdots & -m_{k+1,k} & \ddots & \ddots & & \vdots \\ \vdots & & \vdots & \vdots & 0 & \ddots & \ddots & 0 \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -m_{n,k} & 0 & \cdots & 0 & 1 \end{bmatrix}$$

to obtain

$$A^{(k+1)}\mathbf{x} = M^{(k)}A^{(k)}\mathbf{x} = M^{(k)}\cdots M^{(1)}A\mathbf{x} = M^{(k)}\mathbf{b}^{(k)} = \mathbf{b}^{(k+1)} = M^{(k)}\cdots M^{(1)}\mathbf{b}. \quad (6.9)$$

The process ends with the formation of $A^{(n)}\mathbf{x} = \mathbf{b}^{(n)}$, where $A^{(n)}$ is the upper triangular matrix

$$A^{(n)} = \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{n-1,n}^{(n-1)} \\ 0 & \cdots & 0 & a_{nn}^{(n)} \end{bmatrix},$$

given by

$$A^{(n)} = M^{(n-1)}M^{(n-2)}\cdots M^{(1)}A.$$